Noncommutative complex analysis and
Bargmann-Segal multipliers

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We state several equivalent noncommutative versions of the Cauchy-Riemann equations and characterize the unbounded operators on $L^2(\mathbf{R})$ which satisfy them. These operators arise from the creation operator via a functional calculus involving a class of entire functions, identified by Newman and Shapiro ([13], [14]), which act as unbounded multiplication operators on Bargmann-Segal space.

1. Noncommutative Cauchy-Riemann equations.

Various branches of the field of operator algebras are, following Connes [6], Effros [7], Woronowicz [17], and others, often viewed as "noncommutative" or "quantized" versions of classical mathematical subjects. Underlying this attitude is a rather diverse family of fundamental examples, of which the noncommutative tori [15] and the noncommutative plane ([11], [12]) are probably the most basic. Our aim here is to determine an appropriate version of complex structure on the latter.

The basic elements of the noncommutative plane are these. The underlying Hilbert space is $L^2(\mathbf{R})$. Compact operators play the role of continuous functions which vanish at infinity and bounded operators play the role of bounded measurable functions. The unbounded self-adjoint operators $Q = M_x$ (multiplication by x) and $P = -i\hbar\frac{d}{dx}$, where $\hbar$ is a nonzero real number, are analogous to the coordinate functions, and commutation with the operators $-\frac{1}{\hbar}P$ and $\frac{1}{\hbar}Q$ correspond to the partial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

Physically, this structure characterizes a spinless, nonrelativistic, one-dimensional quantum-mechanical particle. The "noncommutative plane" intuition is motivated by the fact that the phase space of a classical one-dimensional particle is the ordinary plane, the classical position and momentum observables are the coordinate functions on $\mathbf{R}^2$, and so on.

In what sense does the noncommutative plane carry a complex structure? Consider a differentiable function $\phi : \mathbf{C} \to \mathbf{C}$. By regarding it as a function from $\mathbf{R}^2$ into $\mathbf{C}$, we may condense the Cauchy-Riemann equations, which diagnose whether ϕ is holomorphic, into the single condition

$$\frac{\partial\phi}{\partial y} = i\frac{\partial\phi}{\partial x}.$$

In terms of the unbounded multiplication operator $M_\phi : f \to \phi f$ acting on $L^2(\mathbf{R}^2)$, this condition can formally be reexpressed as

$$[M_\phi, \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}] = 0$$

or equivalently

$$[M_\phi, \frac{\partial}{\partial y} - i\frac{\partial}{\partial x}] = 0.$$

(The general principle is that $[M_\phi, \frac{\partial}{\partial x}] = -M_{\partial\phi/\partial x}$, etc.)

Now the noncommutative analogs of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are commutation with $-\frac{1}{\hbar}P$ and $\frac{1}{\hbar}Q$. So, depending on whether we identify position with the x-axis and momentum with the y-axis or vice versa, the noncommutative version of the Cauchy-Riemann equations is either the condition

$$[A, a^-] = 0 \tag{$*$}$$

or the condition

$$[A, a^+] = 0, \tag{$**$}$$

where a^- is the annihilation operator $a^- = \frac{1}{2}(Q + iP)$ and a^+ is the creation operator $a^+ = \frac{1}{2}(Q - iP)$. Either condition is to be satisfied by a possibly unbounded operator A acting on $L^2(\mathbf{R})$.

In classical complex analysis the variables can be switched without essentially changing the theory, but in the noncommutative setting this is not true: the creation and annihilation operators are not equivalent. However, if one uses the right domains they are each other's adjoints, so the class of operators which satisfy $(*)$ ought to be adjoint to the class of operators which satisfy $(**)$.

This is not a rigorous assertion since $(*)$ and $(**)$ are only formal expressions; there is no standard way to define the commutator of two unbounded operators. To make these conditions precise we need a clearer picture of the operators a^+ and a^-. Mathematically, the nicest representation of the creation and annihilation operators — and the setting in which their natural domains are apparent — is on Bargmann-Segal space, so we go there to pursue the question further.

2. Bargmann-Segal formulation.

From now on we set $r = 1/\hbar$ and assume $r > 0$. (Changing the sign of $\hbar$ is tantamount to interchanging P and Q.) The Bargmann-Segal space, also known as Bargmann-Fock or Fischer space, is the Hilbert space $\mathcal{F}$ consisting of all entire functions on the complex plane which are square-integrable with respect to the Gaussian measure μ defined by $d\mu(z) = (r/\pi)e^{-r|z|^2}dz$. It is isomorphic to $L^2(\mathbf{R})$ via the Bargmann-Segal transform, which takes the natural basis of $\mathcal{F}$ — the orthogonal functions z^n, appropriately normalized — to the Hermite basis of $L^2(\mathbf{R})$. Under this transformation the operators a^+ and a^- on $L^2(\mathbf{R})$ are respectively identified with the operators M_z and $\frac{1}{r}\frac{d}{dz}$ on $\mathcal{F}$. The natural domain of M_z is $\{f \in \mathcal{F} : zf \in \mathcal{F}\}$ and the corresponding domain of $\frac{1}{r}\frac{d}{dz} = M_z^*$ is $\{f \in \mathcal{F} : \bar{z}f \in L^2(\mathbf{C}, d\mu)\}$ (see e.g. [5]). See [8] for a thorough treatment of the Bargmann-Segal transform and see [2], [9], [10] for work on Toeplitz operators on $\mathcal{F}$.

Let $\mathcal{K}$ be the linear span in $\mathcal{F}$ of the reproducing kernels $e_w(z) = e^{rz\bar{w}}$ ($w \in \mathbf{C}$). These functions have the property that $\langle f, e_w \rangle = f(w)$ for all $f \in \mathcal{F}$. Let $\mathcal{PK}$ be the

algebra generated by $\mathcal{K}$ together with the function z. Also, let Λ be the "operator class" of Newman and Shapiro ([13], [14]); this consists of those entire functions ϕ which satisfy $|\phi(z)| = O(e^{r|z|^2/2 - N|z|})$ for all $N > 0$. The following result is given in the unpublished manuscript [14]; for the reader's convenience we include the proof.

Theorem 1. ([14], § 2.2.) *Let ϕ be an entire function on $\mathbf{C}$. Then $\phi \in \Lambda$ if and only if $\phi e_w \in \mathcal{F}$ for all $w \in \mathbf{C}$. The class Λ is closed under differentiation.*

Proof. The forward direction of the first assertion is a routine calculation. For the reverse direction, suppose $\phi e_w \in \mathcal{F}$ for all w and fix $N > 0$. Let $w_k = (\sqrt{2}N/r)e^{2\pi i(k/4)}$ for $0 \leq k \leq 3$. Then

$$|\phi(z)e^{rz\bar{w}_k}| = |\langle \phi e_{w_k}, e_z \rangle| \leq \|\phi e_{w_k}\|e^{r|z|^2/2}$$

for each k, by the Cauchy-Schwartz inequality. Also, for any $z \in \mathbf{C}$ there is a value of k such that $-\pi/4 \leq \operatorname{Arg}(z\bar{w}_k) \leq \pi/4$, and for this k we have $\operatorname{Re}(z\bar{w}_k) \geq |z\bar{w}_k|/\sqrt{2} = N|z|/r$. Thus

$$|\phi(z)e^{rz\bar{w}_k}| = |\phi(z)|e^{r\operatorname{Re}(z\bar{w}_k)} \geq |\phi(z)|e^{N|z|}$$

and we conclude that

$$|\phi(z)| \leq C \cdot e^{r|z|^2/2 - N|z|}$$

where $C = \max_{0 \leq k \leq 3}\|\phi e_{w_k}\|$. This shows that $\phi \in \Lambda$.

Closure of Λ under differentiation follows from the formula

$$\phi'(w) = \langle \frac{d}{dz}\phi, e_w \rangle = \langle \phi, rze_w \rangle$$
$$= (r^2/\pi) \int \bar{z}\phi(z)e^{r(\bar{z}w - |z|^2)}dz;$$

the bound $|\phi(z)| = O(e^{r|z|^2/2 - (N+1)|z|})$ then yields

$$|\phi'(w)| \leq C \int |z|e^{r|z|^2/2 - (N+1)|z|} \, e^{r(\operatorname{Re}(z\bar{w}) - |z|^2)} \, dz$$
$$= C \int |z|e^{r|w|^2/2}e^{-r|z-w|^2/2} \, e^{-(N+1)|z|} \, dz$$
$$= Ce^{r|w|^2/2 - N|w|} \int |z|e^{-r|z-w|^2/2 - N(|z|-|w|)} \, e^{-|z|} \, dz.$$

The final integral is finite because $|z - w| \geq \big||z| - |w|\big|$, so that the first exponent in the integrand is bounded above. Thus $\phi'(w) = O(e^{r|w|^2/2 - N|w|})$, as desired. ∎

We remark that Λ is also closed under integration; this follows from an elementary estimate based on the inequality

$$|\phi(z)| \leq |\phi(0)| + \int_0^{|z|} |\phi'(t\frac{z}{|z|})|dt.$$

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For $\phi \in \Lambda$, let M_ϕ be the operator of multiplication by ϕ, with domain $D(M_\phi) = \{g \in \mathcal{F} : \phi g \in \mathcal{F}\}$. Thus $\mathcal{K} \subset D(M_\phi)$ by the preceding result, and in fact $\mathcal{PK} \subset D(M_\phi)$ since every function in $\mathcal{PK}$ is of exponential type, i.e. is $O(e^{N|z|})$ for some N. The adjoint M_ϕ^* has similar properties:

Proposition 2. *Let $\phi \in \Lambda$. Then $\mathcal{PK} \subset D(M_\phi^*)$, $\mathcal{K}$ is a core for M_ϕ^*, and we have $M_\phi^*(e_w) = \bar{\phi}(w)e_w$ for all $w \in \mathbf{C}$.*

Proof. For $w \in \mathbf{C}$ and $g \in D(M_\phi)$ we have

$$\langle M_\phi g, e_w \rangle = \phi(w)g(w) = \langle g, \bar{\phi}(w)e_w \rangle.$$

This shows that $\mathcal{K} \subset D(M_\phi^*)$ and $M_\phi^*(e_w) = \bar{\phi}(w)e_w$.

Let A be the closure of $M_\phi^*|_\mathcal{K}$. To prove that $\mathcal{K}$ is a core for M_ϕ^*, we must show that $M_\phi^* \subset A$, or equivalently that $A^* \subset M_\phi$. Let $g \in D(A^*)$ and let $h = A^*g$. Then we have

$$h(w) = \langle h, e_w \rangle = \langle g, Ae_w \rangle = \langle g, \bar{\phi}(w)e_w \rangle = \phi(w)g(w)$$

for all $w \in \mathbf{C}$. Thus $\phi g = h \in \mathcal{F}$, hence $g \in D(M_\phi)$, and so $A^* \subset M_\phi$ as desired.

Finally, $\mathcal{PK} \subset D(M_\phi^*)$ follows from the standard fact that $D(M_\phi^*)$ contains $\{f \in \mathcal{F} : \bar{\phi}f \in L^2(\mathbf{C}, d\mu)\}$. If $f \in \mathcal{PK}$ then $\bar{\phi}f$ satisfies the same growth condition as ϕ and hence is square-integrable, so this implies $f \in D(M_\phi^*)$. ∎

We will argue in the next section that the operators M_ϕ for $\phi \in \Lambda$ are the natural solutions to $(**)$ and their adjoints are the natural solutions to $(*)$.

3. The main result.

The following seems to be a reasonable notion of commutation for unbounded operators. In our setting the classes $\mathcal{K}$ and $\mathcal{PK}$ will play the role of E.

Definition 3. Let E be a linear subspace of a Hilbert space H and let A and B be unbounded operators on H both of whose domains contain E, that is $E \subset D(A) \cap D(B)$. Suppose also that $E \subset D(A^*) \cap D(B^*)$. Then we say that A and B *commute relative to* E and write $[A, B] = 0$ (rel E) if

$$\langle A\eta, B^*\xi \rangle = \langle B\eta, A^*\xi \rangle$$

for all $\eta, \xi \in E$. ∎

We now present our main theorem. We regard conditions (a) and (b) as justifying the idea that operators which satisfy these conditions are the natural solutions to equation $(**)$: (b) is the stronger version, which says that A commutes with M_z relative to the

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"large" domain $\mathcal{PK}$, while (a) involves the "small" domain $\mathcal{K}$ and the morally weaker condition that A commutes with M_{e_w} for all $w \in \mathbf{C}$. Likewise, conditions (f) and (g) are weak and strong versions of equation $(*)$.

Theorem 4. *Let A be a closed, densely defined, unbounded operator on $\mathcal{F}$. Suppose $\mathcal{K} \subset D(A) \cap D(A^*)$ and K is a core for A^*. Then the following are equivalent:*

(a) $[A, M_{e_w}] = 0$ (rel $\mathcal{K}$) for all $w \in \mathbf{C}$;
(b) $\mathcal{PK} \subset D(A) \cap D(A^*)$ and $[A, M_z] = 0$ (rel $\mathcal{PK}$);
(c) $A = M_\phi$ for some $\phi \in \Lambda$;
(d) $A^* e_w = \bar{\phi}(w) e_w$ for some $\phi \in \Lambda$ and all $w \in \mathbf{C}$;
(e) e_w is an eigenvector of A^* for all $w \in \mathbf{C}$;
(f) $[\frac{d}{dz}, A^*] = 0$ (rel $\mathcal{K}$); and
(g) $\mathcal{PK} \subset D(A) \cap D(A^*)$ and $[\frac{d}{dz}, A^*] = 0$ (rel $\mathcal{PK}$).

Proof.

(a) $\Rightarrow$ (c). Suppose (a) holds and let $\phi = A(e_0) = A(1)$. Then for any $v, w \in \mathbf{C}$ we have

$$\langle A e_v, e_w \rangle = \langle M_{e_v} e_0, A^* e_w \rangle = \langle A e_0, M_{e_v}^* e_w \rangle = e_v(w) \langle \phi, e_w \rangle = \phi(w) e_v(w);$$

this implies that $A e_v = \phi e_v$. Hence $\phi e_v \in \mathcal{F}$ for all v, so $\phi \in \Lambda$ follows from Theorem 1. Also $A|_{\mathcal{K}} = M_\phi|_{\mathcal{K}}$.

Let $A_0 = A|_{\mathcal{K}}$. Then the above shows that $A_0 \subset M_\phi$ and therefore $M_\phi^* \subset A_0^*$. But also $A^* \subset A_0^*$, and $\mathcal{K}$ is a core for both A^* (by hypothesis) and M_ϕ^* (by Proposition 2). So $A^* = M_\phi^*$ and hence $A = M_\phi$.

(c) $\Rightarrow$ (d). This was shown in Proposition 2.

(d) $\Rightarrow$ (e). Vacuous.

(e) $\Rightarrow$ (a). Suppose every e_w is an eigenvector of A^* and fix $u, v, w \in \mathbf{C}$. Let $A^* e_w = \bar{\lambda} e_w$. Then
$$\langle M_{e_u} e_v, A^* e_w \rangle = \lambda \langle e_{u+v}, e_w \rangle = \lambda e^{r(\bar{u}+\bar{v})w}$$
while

$$\langle A e_v, M_{e_u}^* e_w \rangle = \langle A e_v, e^{ru\bar{w}} e_w \rangle = e^{r\bar{u}w} \langle e_v, A^* e_w \rangle = \lambda e^{r\bar{u}w} \langle e_v, e_w \rangle = \lambda e^{r\bar{u}w} e^{r\bar{v}w}.$$

So $[A, M_{e_u}] = 0$ (rel $\mathcal{K}$).

(c) $\Rightarrow$ (b). $\mathcal{PK} \subset D(M_\phi)$ is easy and $\mathcal{PK} \subset D(M_\phi^*)$ was shown in Proposition 2. If $\eta, \xi \in \mathcal{PK}$ then
$$\langle M_\phi \eta, M_z^* \xi \rangle = \langle \phi \eta, M_z^* \xi \rangle = \langle z \phi \eta, \xi \rangle$$
and

$$\langle M_z \eta, M_\phi^* \xi \rangle = \langle z \eta, M_\phi^* \xi \rangle = \langle \phi z \eta, \xi \rangle.$$

So $[M_\phi, M_z] = 0$ (rel $\mathcal{PK}$).

(b) $\Rightarrow$ (g). This is immediate from the fact that $M_z^* = \frac{1}{r}\frac{d}{dz}$ and the definition of commutation.

(g) $\Rightarrow$ (f). Vacuous.

(f) $\Rightarrow$ (e). Suppose $[\frac{d}{dz}, A^*] = 0$ (rel $\mathcal{K}$). Then for any $v, w \in \mathbf{C}$

$$\langle A^* e_v, z e_w \rangle = \langle \frac{1}{r}\frac{d}{dz} e_v, A e_w \rangle = \langle \bar{v} e_v, A e_w \rangle = \langle A^* e_v, v e_w \rangle.$$

It follows that $A^* e_v$ is orthogonal to $(z - v)f$ if f is any finite linear combination of the functions e_w.

In particular let

$$f_h = \frac{1}{h^n} \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} e_{kh}.$$

Then f_h is an nth order difference quotient and it converges in $\mathcal{F}$ to the nth Hermite polynomial H_n as $h \to 0$. We also have $(z - v)f_h \to (z - v)H_n$ in $\mathcal{F}$. As the H_n span all polynomials, it follows that Ae_v is orthogonal to every polynomial which vanishes at v. So letting $\lambda = \langle Ae_v, e_v \rangle / \|e_v\|^2$, we have that $Ae_v - \lambda e_v$ is orthogonal to every polynomial. Hence $Ae_v = \lambda e_v$, as desired. ∎

Thus, if we fix the convention that $(**)$ is the noncommutative version of the Cauchy-Riemann equations, then the unbounded operators which play the role of "noncommutative" entire functions are the operators M_ϕ ($\phi \in \Lambda$) — which is somewhat surprising considering that these operators all commute. Noncommutativity resides in the fact that they are not normal.

In any reasonable sense $M_\phi = \phi(M_z)$, so these are just the operators that one obtains via functional calculus by applying functions in the operator class Λ to the creation operator. Thus this class of operators possesses natural linear, algebraic, and topological structures isomorphic to those on Λ.

Remark 5. It is also possible to differentiate these operators with respect to the two noncommutative coordinates, in the sense of taking the commutator with $-\frac{1}{\hbar}P$ and $\frac{1}{\hbar}Q$. Since $Q = (a^+ + a^-)$ and $P = i(a^+ - a^-)$, in the Bargmann-Segal representation these operators are given by $Q = (M_z + \frac{1}{r}\frac{d}{dz})$ and $P = i(M_z - \frac{1}{r}\frac{d}{dz})$. So if $\phi \in \Lambda$ and $\eta \in \mathcal{F}$ is of exponential type then

$$
\begin{aligned}
[\frac{1}{\hbar}Q, M_\phi]\eta &= \frac{1}{\hbar}Q(M_\phi\eta) - M_\phi(\frac{1}{\hbar}Q\eta) \\
&= r(Q(\phi\eta) - \phi \cdot Q\eta) \\
&= r(z\phi\eta + \frac{1}{r}(\phi'\eta + \phi\eta') - z\phi\eta - \frac{1}{r}\phi\eta') \\
&= M_{\phi'}\eta
\end{aligned}
$$

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and

$$[-\frac{1}{\hbar}P, M_\phi]\eta = -\frac{1}{\hbar}P(M_\phi\eta) - M_\phi(-\frac{1}{\hbar}P\eta)$$

$$= -r(P(\phi\eta) - \phi \cdot P\eta)$$

$$= -ir(z\phi\eta - \frac{1}{r}(\phi'\eta + \phi\eta') - z\phi\eta + \frac{1}{r}\phi\eta')$$

$$= M_{i\phi'}\eta.$$

Thus differentiation of M_ϕ in the noncommutative sense corresponds to differentiation of the symbol ϕ, and we have also verified that M_ϕ satisfies the Cauchy-Riemann condition in the form

$$[-\frac{1}{\hbar}P, M_\phi] = i[\frac{1}{\hbar}Q, M_\phi].$$

Remark 6. We can also identify a reasonable noncommutative version of harmonic functions. Here the classical condition is $\frac{d^2\phi}{dzd\bar{z}} = 0$, which we can formally quantize as

$$[a^+, [a^-, A]] = [M_z, [M_z^*, A]] = 0. \tag{$\dagger$}$$

The simplest way to make this harmonicity condition precise is to invoke Theorem 4 to interpret it as asserting that $[M_z^*, A] = M_\phi$ for some $\phi \in \Lambda$, and then to require, minimally, that

$$M_z^*Ae_v - AM_z^*e_v = M_\phi e_v$$

for all $v \in \mathbf{C}$. (In particular, we ask that $Ae_v \in D(M_z^*)$ for all $v \in \mathbf{C}$; a short computation shows that this is equivalent to the requirement that $e_v \in D([M_z^*, A])$ in the sense that the map $\eta \mapsto \langle Ae_v, M_z\eta \rangle - \langle M_z^*e_v, A^*\eta \rangle$ is bounded on $\mathcal{K}$.)

On $\mathcal{K}$, the solutions to ($\dagger$) are precisely those of the form $A = M_\Phi + B$ where $\Phi' = \phi$ and B satisfies $[M_z^*, B] = 0$ (rel $\mathcal{K}$). So by Theorem 4 (f), $B|_\mathcal{K} = M_\Psi^*|_\mathcal{K}$ for some $\Psi \in \Lambda$. Thus, ($\dagger$) implies that $A = M_\Phi + M_\Psi^*$ on $\mathcal{K}$, for some $\Phi, \Psi \in \Lambda$.

In fact, if we assume $\mathcal{K} \subset D(A) \cap D(A^*)$ then $\mathcal{K}$ must be a core for A. For letting A_1 be the closure of $(M_\Phi + M_\Psi^*)|_\mathcal{K}$ and A_2 the closure of $(M_\Phi^* + M_\Psi)|_\mathcal{K}$, we have $A_2 \subset A_1^*$ and $A_1 \subset A_2^*$, hence $A_1^* = A_2$; so $A_2 \subset A^*$ implies $A \subset A_1$. We conclude that reasonable domain restrictions imply that "noncommutative harmonic operators" have $\mathcal{K}$ as a core and are of the form $M_\Phi + M_\Psi^*$ ($\Phi, \Psi \in \Lambda$) on $\mathcal{K}$. Conversely, a short computation shows that any operator of this form does satisfy ($\dagger$).

If we instead begin with $[M_z^*, [M_z, A]] = 0$ as the defining condition of harmonicity, taking adjoints yields $[M_z, [M_z^*, A^*]] = 0$, so that (again, up to reasonable domain assumptions) A^* is of the form $M_\Phi + M_\Psi^*$, and hence so is A.

4. Counterexamples.

The requirement that $\mathcal{K}$ be a core for A^* in Theorem 4 is probably unnecessary. It was conjectured in [13] (see also [4], [5]) that $\mathcal{K}$ is a core for every M_ϕ ($\phi \in \Lambda$), and if this

is true then our hypothesis that $\mathcal{K}$ is a core for A^* is superfluous. For the only place that it is used is in the proof of (a) $\Rightarrow$ (c), and if we knew that $\mathcal{K}$ was a core for M_ϕ we could reason instead as follows: we know $A|_\mathcal{K} = M_\phi|_\mathcal{K}$, hence $M_\phi \subset A$ since $\mathcal{K}$ is a core for M_ϕ; therefore $A^* \subset M_\phi^*$, but $\mathcal{K} \subset D(A^*)$ and $\mathcal{K}$ is also a core for M_ϕ^*, hence $A^* = M_\phi^*$ and so $A = M_\phi$.

On the other hand, the requirement $\mathcal{K} \subset D(A) \cap D(A^*)$ is crucial. There are operators for which this is not true but which do commute with M_z is a reasonable sense, namely any multiplication operator M_ϕ with $\phi \notin \Lambda$.

Moving to this broader setting introduces a number of pathologies. First, observe that since M_z is unbounded there must exist $f \in \mathcal{F}$ such that $zf \notin \mathcal{F}$. Then $1 \in D(M_f)$ but $z \notin D(M_f)$. In fact, using the fact that the norm of z^n in $\mathcal{F}$ is $r^{n/2}\sqrt{n!}$ it is easy to check that the function $f(z) = \sum a_n z^n$ with $a_n = (r^n n! (n+1)^2)^{-1/2}$ has this property, for example. Furthermore, the function $g(z) = \sum a_{n+k} z^n$ satisfies $pg \in \mathcal{F}$ if p is a polynomial of degree at most k, and $pg \notin \mathcal{F}$ if p is a polynomial of degree larger than k.

Note that in general if $D(M_\phi)$ contains any polynomial p of degree k then it contains all polynomials of degree $\leq k$. This follows immediately from the definitions and the fact that at ∞, $|p| \sim c|r|^{2n}$.

In fact, given any nonnegative integer k it is possible to find an entire ϕ so that $D(M_\phi)$ consists exactly of $\mathcal{P}_k = \{f : f$ is a polynomial of degree $\leq k\}$. A convenient way to see this is to follow the lead of Seip and Wallstein. In Proposition 2.1 of [16] they show, in our language, that if ϕ is the Weierstrass σ-function with period lattice $\{\sqrt{\frac{\pi}{r}}(n + im) : n, m \in \mathbf{Z}\}$ then $D(M_\phi) = \{0\}$. Suppose now that k is given. Select any $k + 2$ of the zeros of ϕ and let p be a polynomial with exactly those points as its (simple) zeros. The natural modification of the argument in [16] gives

$$D(M_{\phi/p}) = \left\{ f : f \text{ entire and } \int_{|z|>1} |f|^2 |z|^{-2(k+2)} dz < \infty \right\}.$$

This last set is $\mathcal{P}_k$, as required. (We note, as is pointed out in [16], that although this particular choice of ϕ is extremely convenient computationally it is not essential for such an argument.)

It seems reasonable to restrict attention to the case that M_ϕ has dense domain, but there are still serious problems in this case. For any functions f and g of exponential type, any $w \in \mathbf{C}$ with $|w| < 1$, and any $a \in (1 - \frac{1}{2|w|}, \frac{1}{2|w|})$ we have $ge^{-awz^2} \in \mathcal{F}$ and

$$(fe^{wz^2})(ge^{-awz^2}) = fge^{(1-a)wz^2} \in \mathcal{F}$$

since $|-aw| < \frac{1}{2}$ and $|(1-a)w| < \frac{1}{2}$. It follows that M_ϕ, where $\phi = fe^{wz^2}$, has a "large" domain, but if $|w| \geq \frac{1}{2}$ then nothing in its domain has exponential type, and in particular $\mathcal{P}\mathcal{K}$ is disjoint from $D(M_\phi)$. From a comment on page 126 of [3] it appears that there are similar examples of such ϕ of order 2 and arbitrarily large type, such that $D(M_\phi)$ contains functions of order 2 and type less than $\frac{1}{2}$. Another interesting example is given in [4]; this

is a function ϕ such that every polynomial belongs to $D(M_\phi)$ but $\mathcal{K} \not\subset D(M_\phi)$ and the polynomials are not a core of M_ϕ.

For these reasons, from a purely operational point of view it clearly is desirable to require that $\mathcal{K} \subset D(A) \cap D(A^*)$. Some philosophical motivation may also come from the view that the coherent states e_w play the role of the points of the noncommutative plane (see e.g. [1]).

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